Worcester County Mathematics League

Varsity Meet 2 - November 20, 2019

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League
Varsity Meet 2 - November 20, 2019
Answer Key

Round 1 - Fractions, Decimals, and Percents

1. 37.5% or 37.5 percent
2. $\frac{13}{8}$ or $1\frac{5}{8}$ or 1.625
3. $\frac{7}{12}$

Round 2 - Algebra I

1. 39
2. 9 and 11 (any order)
3. $(2, 0)$ and $(5, 0)$ (either order)

Round 3 - Parallel Lines and Polygons

1. $x = 12$
2. 96° or 96 degrees
3. $4\sqrt{5}$

Round 4 - Sequences and Series

1. -19 (note the negative)
2. 8 and 17 (either order)
3. 7500

Round 5 - Matrices and Systems of Equations

1. $y = 83$
2. $\begin{bmatrix} 6 & 1 \\ 9 & 4 \end{bmatrix}$
3. $\left(\frac{9}{5}, \frac{16}{5}\right), \left(-\frac{9}{5}, -\frac{16}{5}\right)$ (both required)

Team Round

1. $\frac{63}{235}$
2. 10
3. $8\sqrt{3}$
4. 3000
5. -6
6. 195
7. $\frac{1}{4}$
8. 51
9. $\frac{8}{27}$

Worcester County Mathematics League
Varsity Meet 2 - November 20, 2019
Round 1 - Fractions, Decimals, and Percents

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. An item for sale is marked up 25%, then reduced in price by 50%, and later reduced a further 40%. What percentage of the original sales price is the current sales price?

2. Simplify:
$$\left(\frac{5}{18}+\frac{2}{9}+\frac{1}{16}+\frac{1}{4}\right)\div\frac{1}{2}$$

3. Four points, A, B, C, and D, lie on a line in that order such that $AB=\frac{1}{5}BD$ and $BC=\frac{3}{5}AC$. Determine the value of $\frac{CD}{AD}$ and express your answer as a fraction in simplest form.

ANSWERS

(1 pt) 1. ________________ percent

(2 pts) 2. ________________

(3 pts) 3. ________________ Bromfield, Douglas, AMSA

Worcester County Mathematics League
Varsity Meet 2 - November 20, 2019
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. The United States earned 103 medals at the 2004 Summer Olympics in Athens. The number of silver medals earned was 4 more than the number of gold medals. The number of bronze medals earned 6 less than the number of gold medals. How many silver medals were earned?

2. The sum of the squares of two consecutive positive odd integers is 202. Find the integers.

3. Three lines lie in the xy-plane such that one pair of lines intersects at $(-1, 3)$ and another pair of lines intersects at $(8, 6)$. Find all possible intersection points of the third pair of lines if the two lines are perpendicular and have the same x-intercept.

ANSWERS

(1 pt) 1. ________________

(2 pts) 2. ________________

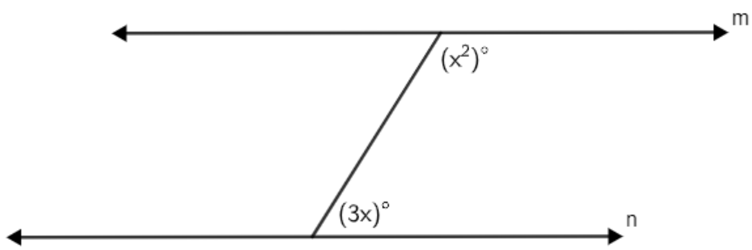
(3 pts) 3. ________________

Wachusett, Ashland, QSC

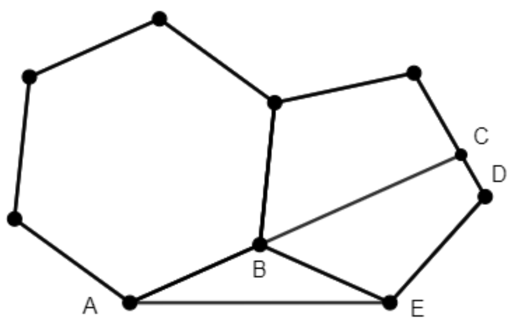
Worcester County Mathematics League
Varsity Meet 2 - November 20, 2019
Round 3 - Parallel Lines and Polygons

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

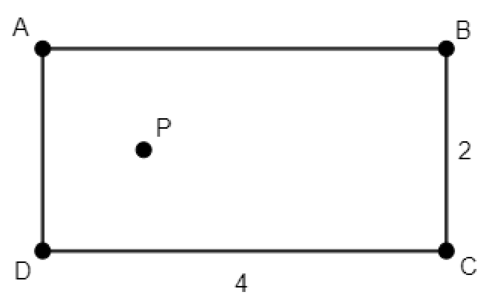
1. Find the value of x that causes lines m and n to be parallel.



2. The hexagon and the pentagon are regular. Find the measure of $\angle DCA$.



3. Consider rectangle $ABCD$ with length 4 units and width 2 units. Point P is located one unit to the right and one unit above point D (i.e. if D is $(0, 0)$ then P is $(1, 1)$). Parallelogram $WXYZ$ is placed inside the rectangle such that points W, X, Y, and Z lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. If $\overline{YZ}$ contains P and $\Delta ZDY \sim \Delta ZAW$, find the perimeter of $WXYZ$.



ANSWERS

(1 pt) 1. $x =$ ____________________

(2 pts) 2. ____________________ degrees

(3 pts) 3. ____________________ NDA, Worcester Academy, Leicester

Worcester County Mathematics League
Varsity Meet 2 - November 20, 2019
Round 4 - Sequences and Series

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $2x-13$, $3x-10$, and $x+8$ are third, fourth, and fifth terms in an arithmetic sequence, find the first term.

2. Integers a and b are the first two terms of a Fibonacci-like sequence in which each term after b is the sum of the previous two terms. If the fourth term of the sequence is 42 and the sixth term is 109, find the value of a and b.

3. Find the sum of the first 100 terms of:

$$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, \ldots$$

ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

QSC, Worcester Academy, Tahanto

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. If $\begin{bmatrix} 43 & 2m+10 \\ 5x-2 & 3m-8 \end{bmatrix} = \begin{bmatrix} z+3 & z \\ y & 2x+3 \end{bmatrix}$, find y.

2. Solve

$$\begin{bmatrix} 6 & -3 \\ 5 & -2 \end{bmatrix} X = \begin{bmatrix} 9 & -6 \\ 12 & -3 \end{bmatrix}$$

for the matrix X.

3. Determine all ordered pairs (x, y) such that

$$x^2 + xy = 9 \qquad \text{and} \qquad y^2 + xy = 16$$

.

ANSWERS

(1 pt) 1. $y =$ ______________________________

(2 pts) 2. $X = \begin{bmatrix} \_\_\_\_\_ & \_\_\_\_\_ \\ \_\_\_\_\_ & \_\_\_\_\_ \end{bmatrix}$

(3 pts) 3. ____

Southbridge, Sutton, Wachusett

Worcester County Mathematics League
Varsity Meet 2 - November 20, 2019
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Calculate 30% of
$$\frac{\frac{4}{9}-\frac{1}{18}}{\frac{11}{36}+\frac{7}{54}}$$
as a reduced fraction.

2. How many lattice points (m, n) (points in the xy-plane with integer coefficients) are on the line segment joining $(-2, 3)$ and $(34, 30)$?

3. In equilateral triangle ABC with each side of length 6, a segment is drawn parallel to one side to form a trapezoid with legs of length 4. Find the area of the trapezoid.

4. Milo lists the first 100 positive integers, then crosses off every integer divisible by 4 or 5. What is the sum of the remaining integers?

5. Find xyz if
$$\begin{cases} 2x + 2y + z = 1 \\ x - y + 6z = 21 \\ 6x + 4y - 2z = -8 \end{cases}$$

6. How many distinct three-letter words can be created from the letters in WOCOMAL?

7. Evaluate the following expression if $x = 6$ and $y = 2$.
$$\frac{x^4 + x^3 - 6x^2}{xy^3 - 2y^3} \div \frac{x^2 + 8x + 15}{x^2 + 2x - 15} \cdot \frac{y^2}{x^3}$$

8. There are n concentric circles with radii r_1, r_2, r_3, $\ldots$, r_n and areas A, $2A$, $3A$, $\ldots$, nA, respectively. Find the smallest value of n where r_n is less than 1% larger than r_{n-1}.

9. Mr. Smithers comes home from the grocery store with three bags of groceries. Each bag has three distinct food items:
   - Bag 1: Vegetable, pasta, meat.
   - Bag 2: Vegetable, vegetable, meat.
   - Bag 3: Pasta, pasta, meat.

   He picks one item at random from each bag. What is the probability that he has picked one of each type of food (vegetable, pasta, meat)?

Worcester County Mathematics League
Varsity Meet 2 - November 20, 2019
Team Round Answer Sheet

ANSWERS

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

7. ______________________________

8. ______________________________

9. ______________________________

Southbridge, Mass Academy, Quaboag, Worcester Academy, Shrewsbury, Bancroft, Sutton, QSC, QSC

Worcester County Mathematics League
Varsity Meet 2 - November 20, 2019
Answer Key

Round 1 - Fractions, Decimals, and Percents

1. 37.5% or 37.5 percent
2. $\frac{13}{8}$ or $1\frac{5}{8}$ or 1.625
3. $\frac{7}{12}$

Round 2 - Algebra I

1. 39
2. 9 and 11 (any order)
3. $(2, 0)$ and $(5, 0)$ (either order)

Round 3 - Parallel Lines and Polygons

1. $x = 12$
2. $96°$ or 96 degrees
3. $4\sqrt{5}$

Round 4 - Sequences and Series

1. -19 (note the negative)
2. 8 and 17 (either order)
3. 7500

Round 5 - Matrices and Systems of Equations

1. $y = 83$
2. $\begin{bmatrix} 6 & 1 \\ 9 & 4 \end{bmatrix}$
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Team Round

1. $\frac{63}{235}$
2. 10
3. $8\sqrt{3}$
4. 3000
5. -6
6. 195
7. $\frac{1}{4}$
8. 51
9. $\frac{8}{27}$

Round 1 - Fractions, Decimals, and Percents

1. An item for sale is marked up 25%, then reduced in price by 50%, and later reduced a further 40%. What percentage of the original sales price is the current sales price?

Solution: Let the item for sale have price P.

- Increasing the original price P by 25% means the new price is $\frac{5}{4}P$.
- Reducing this price by 50% means the new price is $\frac{5}{4} \cdot \frac{1}{2}P$.
- Reducing this price by an additional 40% means the new price is $\frac{5}{4} \cdot \frac{1}{2} \cdot \frac{3}{5}P$.

This simplifies to $\frac{5}{4} \cdot \frac{1}{2} \cdot \frac{3}{5}P = \frac{3}{8}P$ meaning that the most recent price is $\frac{3}{8}$, or $\boxed{37.5\%}$, of the original.

2. Simplify:

$$\left(\frac{5}{18} + \frac{2}{9} + \frac{1}{16} + \frac{1}{4}\right) \div \frac{1}{2}$$

Solution: Since the least common denominator for these fractions is 144, we can change this expression to

$$\left(\frac{40}{144} + \frac{32}{144} + \frac{9}{144} + \frac{36}{144}\right) \div \frac{1}{2}.$$

Continuing this sum, we get

$$\left(\frac{117}{144}\right) \cdot 2 = \frac{117}{72} = \boxed{\frac{13}{8} = 1\frac{5}{8} = 1.625}$$

Alternatively,

$$\left(\frac{5}{18} + \frac{2}{9} + \frac{1}{16} + \frac{1}{4}\right) \div \frac{1}{2} = \left(\frac{5}{18} + \frac{4}{18} + \frac{1}{16} + \frac{4}{16}\right) \div \frac{1}{2} = \left(\frac{1}{2} + \frac{5}{16}\right) \cdot 2 = 1 + \frac{5}{8} = \frac{13}{8}.$$

3. Four points, A, B, C, and D, lie on a line in that order such that $AB = \frac{1}{5}BD$ and $BC = \frac{3}{5}AC$. Determine the value of $\frac{CD}{AD}$ and express your answer as a fraction in simplest form.

Solution: Since $AB + BC = AC$ and $BC = \frac{3}{5}AC$, the ratio of AB to BC must be $2:3$.

Additionally, since $AB + BD = AD$ and $AB = \frac{1}{5}BD$, the ratio of AB to BD must be $1:5$ or $2:10$ (to make it compatible with the previous ratio). This leaves the following diagram.

Combining our knowledge, we arrive at

The ratio of CD to AD is therefore $\boxed{\frac{7}{12}}$.

Round 2 - Algebra I

1. The United States earned 103 medals at the 2004 Summer Olympics in Athens. The number of silver medals earned was 4 more than the number of gold medals. The number of bronze medals earned 6 less than the number of gold medals. How many silver medals were earned?

Solution: Let the number of gold medals earned be G, the number of silver medals earned be S, and the number of bronze medals earned be B. We know that

$$G + S + B = G + (G + 4) + (G - 6) = 3G - 2 = 105$$

and we can find that $G = 35$. Therefore, the number of silver medals earned was $S = G + 4 = 35 + 4 = \boxed{39}$ medals.

2. The sum of the squares of two consecutive positive odd integers is 202. Find the integers.

Solution: Let the two consecutive positive odd integers be a and $a + 2$. We know that

$$\begin{aligned} a^2 + (a+2)^2 &= 202 \\ a^2 + a^2 + 4a + 4 &= 202 \\ 2a^2 + 4a - 198 &= 202 \\ a^2 + 2a - 99 &= 0 \\ (a + 11)(a - 9) &= 0 \end{aligned}$$

Since a must be positive, $a = 9$ and $a + 2 = 11$. Our integers are $\boxed{9 \text{ and } 11}$.

3. Three lines lie in the xy-plane such that one pair of lines intersects at $(-1, 3)$ and another pair of lines intersects at $(8, 6)$. Find all possible intersection points of the third pair of lines if the two lines are perpendicular and have the same x-intercept.

Solution: Let the three lines be p, q, and r. Assume p and q intersect at $(-1, 3)$ and q and r intersect at $(8, 6)$. This means that q passes through both points. With the additional information given regarding the x-intercept of p and r (let it be $(a, 0)$), we know the following:

Line	Point 1	Point 2
p	$(-1, 3)$	$(a, 0)$
q	$(-1, 3)$	$(8, 6)$
r	$(a, 0)$	$(8, 6)$

The slope of p and the slope of r must be

$$\text{Slope of } p = m_p = \frac{0 - 3}{a - (-1)} = \frac{-3}{a + 1}$$

$$\text{Slope of } r = m_r = \frac{6 - 0}{8 - a} = \frac{6}{8 - a}$$

and since the lines are perpendicular,

$$m_p \cdot m_r = \frac{-3}{a + 1} \cdot \frac{6}{8 - a} = -1$$

Solving for a,

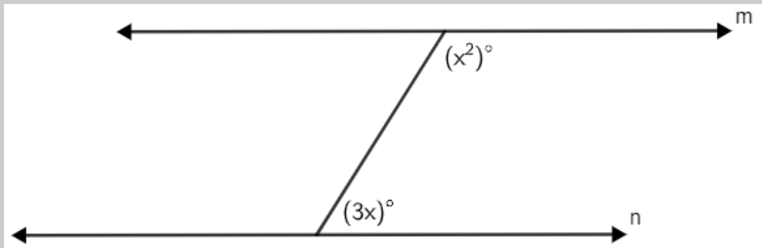
$$\frac{-3}{a+1} \cdot \frac{6}{8-a} = -1$$
$$\frac{-18}{8-a^2+7a} = -1$$
$$-18 = a^2 - 7a - 8$$
$$0 = a^2 - 7a + 10$$
$$0 = (a-2)(a-5)$$

This means the x-intercept of both lines p and r are $\boxed{(2,0) \text{ and } (5,0)}$.

Round 3 - Parallel Lines and Polygons

1. Find the value of x that causes lines m and n to be parallel.

Solution:



In order for two lines to be parallel, the same-side interior angles formed by a transversal must be supplementary.

$$x^2 + 3x = 180$$

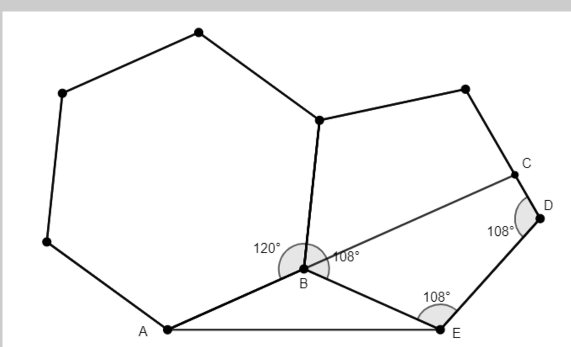
$$x^2 + 3x - 180 = 0$$

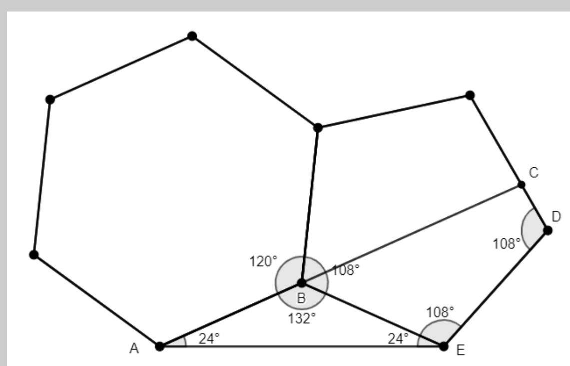
$$(x + 15)(x - 12) = 0$$

Since x must be positive given the angle $(3x)°$, then $\boxed{x = 12}$.

2. The hexagon and the pentagon are regular. Find the measure of $\angle DCA$.

Solution: The interior angles of a regular hexagon are $\dfrac{4 \cdot 180°}{6} = 4 \cdot 30° = 120°$ each and the interior angles of a regular pentagon are $\dfrac{3 \cdot 180°}{5} = 3 \cdot 36° = 108°$ each. We can use this to determine $\measuredangle ABE$ at $132°$ and the base angles of isosceles triangle ΔABE at $24°$ each. Since the interior angles of quadrilateral $ACDE$ must add up to $360°$, $360° - 24° - 24° - 108° - 108° = \boxed{96°}$.





3. Consider rectangle $ABCD$ with length 4 units and width 2 units. Point P is located one unit to the right and one unit above point D (i.e. if D is $(0,0)$ then P is $(1,1)$). Parallelogram $WXYZ$ is placed inside the rectangle such that points W, X, Y, and Z lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively. If $\overline{YZ}$ contains P and $\Delta ZDY \sim \Delta ZAW$, find the perimeter of $WXYZ$.

Solution: Let $DY = a$ and $DZ = b$. Given $AD = 2$ we can say that $AZ = 2 - b$. Similarly, $AW = 4 - a$. Since $\Delta ZDY \sim \Delta ZAW$ we can state that

$$\frac{ZD}{ZA} = \frac{DY}{AW} \Longrightarrow \frac{a}{2-a} = \frac{b}{4-b}$$

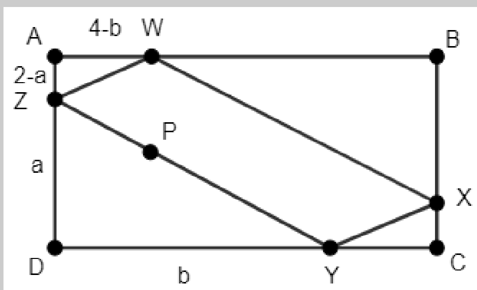
and then

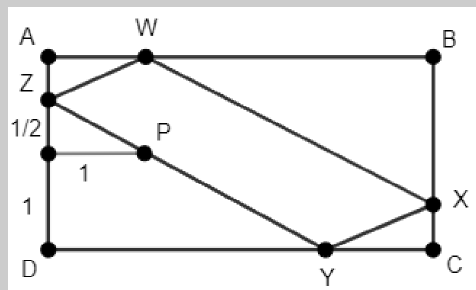
$$4a - ab = 2b - ab$$

$$2a = b$$

$$\frac{a}{b} = \frac{1}{2}.$$

This means $\overline{ZY}$ has slope $\frac{1}{2}$, $DZ = \frac{3}{2}$ (see diagram on right), and $ZA = \frac{1}{2}$. Since $ZD : DY = 1 : 2$, $DY = 3$ and thus $AW = 1$ (similarity).





To find the perimeter of $WXYZ$, we start by finding ZW. Since $ZA^2 + AW^2 = ZW^2$,

$$ZW = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}.$$

With $YX = WZ$ and both WX and ZY being three times YX (similarity), the overall perimeter of $WXYZ$ is $8 \cdot \frac{1}{2}\sqrt{5} = \boxed{4\sqrt{5}}$.

Round 4 - Sequences and Series

1. If $2x - 13$, $3x - 10$, and $x + 8$ are third, fourth, and fifth terms in an arithmetic sequence, find the first term.

Solution: Since $a_4 - a_3 = a_5 - a_4$ for terms in an arithmetic sequence, we have

$$\begin{aligned}(3x-10)-(2x-13) &= (x+8)-(3x-10)\\ x+3 &= -2x+18\\ 3x &= 15\\ x &= 5.\end{aligned}$$

This means

$$a_3 = 2(5) - 13 = -3 \qquad a_4 = 3(5) - 10 = 5 \qquad a_5 = (5) = 8 = 13$$

and the common difference is 8. From the third term (-3) subtract two differences (10) to arrive at $\boxed{a_1 = -19}$.

2. Integers a and b are the first two terms of a Fibonacci-like sequence in which each term after b is the sum of the previous two terms. If the fourth term of the sequence is 42 and the sixth term is 109, find the value of a and b.

Solution: Given the first two terms and the information above,

$$\begin{aligned}a_1 &= a\\ a_2 &= b\\ a_3 &= a+b\\ a_4 &= a+b+b = \underline{a+2b=42}\\ a_5 &= 42+(a+b)\\ a_6 &= a+b+42+42 = \underline{84+a+b=109}\end{aligned}$$

Solving this system for a and b, we begin by noting that $a = 42 - 2b$ and

$$\begin{aligned}84+a+b &= 109\\ 84+42-2b+b &= 109\\ 126-b &= 109\\ b &= 17\end{aligned}$$

Plugging $\boxed{b = 17}$ back into the first equation, we find $a + 2(17) = 42 \Longrightarrow \boxed{a = 8}$.

3. Find the sum of the first 100 terms of:

$$1, 2, 4, 5, 7, 8, 10, 11, 13, 14, \ldots$$

Solution: The sequence contains positive integers that are not multiples of three and can be broken up into two individual sequences:

$$1, 4, 7, 10, \ldots \quad \text{and} \quad 2, 5, 8, 11, \ldots$$

Each sequence must have fifty terms. To find the sum of the first sequence we know

$$S_{50} = \frac{(a_1 + a_{50})50}{2} = \frac{(a_1 + [a_1 + 49d])50}{2} = (1 + [1 + 49(3)])25 = 149(25) = (150 - 1)(25) = 3750 - 25 = 3725.$$

The second sequence begins at 2 and ends at 149:

$$S_{50} = \frac{(a_1 + a_{50})50}{2} = (2 + 149)25 = (151)(25) = 3775.$$

The total of both sequences is $3725 + 3775 = \boxed{7500}$. Alternatively, add them together at the same time:

$$(149)(25) + (151)(25) = 300(25) = 7500.$$

Or, note that the second sequence has 50 terms, each one more than the corresponding terms of the first sequence:

$$3725 + (3725 + 50) = 7500.$$

Or, find the sum of all numbers from 1 to 150 and subtract the multiples of 3 from 3 to 150. Your choice!

Round 5 - Matrices and Systems of Equations

1. If $\begin{bmatrix} 43 & 2m+10 \\ 5x-2 & 3m-8 \end{bmatrix} = \begin{bmatrix} z+3 & z \\ y & 2x+3 \end{bmatrix}$, find y.

Solution: Given that the two matrices are equal, we can write down four separate equations:

$$43 = z+3 \qquad 2m+10 = z$$
$$5x-2 = y \qquad 3m-8 = 2x+3$$

We can identify the value of z first using the top left equation since it's the only equation with one variable ($z = 40$). Substituting into the top right equation, we find that $2m + 10 = 40 \implies m = 15$. Substituting into the bottom right equation, we learn that $3(15) - 8 = 2x + 3 \implies x = 17$. Finally, substituting into the bottom left equation, we find $5(17) - 2 = y \implies \boxed{y = 83}$.

2. Solve

$$\begin{bmatrix} 6 & -3 \\ 5 & -2 \end{bmatrix} X = \begin{bmatrix} 9 & -6 \\ 12 & -3 \end{bmatrix}$$

for the matrix X.

Solution: For the matrix equation $AX = C$ where X is an unknown matrix, $X = A^{-1}C$. We can calculate the inverse of A. First we find the determinant of A as $-12 - (-15) = 3$. Then,

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} -2 & 3 \\ -5 & 6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 & 3 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & 1 \\ -\frac{5}{3} & 2 \end{bmatrix}$$

Multiplying $A^{-1}C$,

$$\begin{bmatrix} -\frac{2}{3} & 1 \\ -\frac{5}{3} & 2 \end{bmatrix} \begin{bmatrix} 9 & -6 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} -6+12 & 4-3 \\ -15+24 & 10-6 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 1 \\ 9 & 4 \end{bmatrix}}$$

Alternatively, recognize that X must be a 2 x 2 matrix to ensure a 2 x 2 product and set up X as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then multiply AX and compare to C:

$$\begin{bmatrix} 6 & -3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ 12 & -3 \end{bmatrix}$$
$$\begin{bmatrix} 6a-3c & 6b-3d \\ 5a-2c & 5b-2d \end{bmatrix} = \begin{bmatrix} 9 & -6 \\ 12 & -3 \end{bmatrix}$$

Two systems can be created to solve for a, b, c, and d to find the unknown matrix.

3. Determine all ordered pairs (x, y) such that

$$x^2 + xy = 9 \qquad \text{and} \qquad y^2 + xy = 16$$

.

Solution: Factor both equations and isolate the common factors:

$$x^2 + xy = 9 \qquad\qquad y^2 + xy = 16$$
$$x(x+y) = 9 \qquad\qquad y(x+y) = 16$$
$$(x+y) = \frac{9}{x} \qquad\qquad (x+y) = \frac{16}{y}$$

Since $\frac{9}{x} = \frac{16}{y}$ we know that $x = \frac{9}{16}y$. Subtracting the two original equations, we see

$$y^2 - x^2 = 7$$

and substituting for x gives us

$$\begin{aligned} y^2 - x^2 &= 7 \\ y^2 - \frac{81}{256}y^2 &= 7 \\ \frac{175}{256}y^2 &= 7 \\ y^2 &= \frac{256 \cdot 7}{175} = \frac{16^2 \cdot 7}{5^2 \cdot 7} \\ y &= \pm\frac{16}{5} \end{aligned}$$

If $y = \pm\frac{16}{5}$, then $x = \pm\frac{9}{5}$. The two points of intersection are $\boxed{\left(\frac{9}{5}, \frac{16}{5}\right) \text{ and } \left(-\frac{9}{5}, -\frac{16}{5}\right)}$.

Alternatively, combine the equations to start:

$$x^2 + 2xy + y^2 = 25$$
$$(x+y)^2 = 25$$
$$x + y = \pm 5$$

Consider the factored equations of

$$x(x+y) = 9 \qquad\qquad y(x+y) = 16$$

and then plug in values. If $x + y = 5$, then

$$x(5) = 9 \qquad\qquad y(5) = 16$$
$$x = \frac{9}{5} \qquad\qquad y = \frac{16}{5}$$

Similarly, if $x + y = -5$,

$$x(-5) = 9 \qquad\qquad y(-5) = 16$$
$$x = -\frac{9}{5} \qquad\qquad y = -\frac{16}{5}$$

and the solution remains the same.

Team Round

1. Calculate 30% of

$$\frac{\frac{4}{9}-\frac{1}{18}}{\frac{11}{36}+\frac{7}{54}}$$

as a reduced fraction.

Solution: Instead of scaling up, scale down: force all individual fractions to have a denominator of 9, then multiply everything by 9:

$$\frac{\frac{4}{9}-\frac{1}{18}}{\frac{11}{36}+\frac{7}{54}} = \frac{\frac{4}{9}-\frac{\frac{1}{2}}{9}}{\frac{\frac{11}{4}}{9}+\frac{\frac{7}{6}}{9}} = \frac{\frac{7}{2}}{\frac{11}{4}+\frac{7}{6}} = \frac{\frac{7}{2}}{\frac{47}{12}} = \frac{7\cdot 12}{2\cdot 47} = \frac{42}{47}.$$

Thirty percent of that value is

$$\frac{3}{10}\cdot\frac{42}{47} = \frac{3}{5}\cdot\frac{21}{47} = \boxed{\frac{63}{235}}.$$

2. How many lattice points (m, n) (points in the xy-plane with integer coefficients) are on the line segment joining $(-2, 3)$ and $(34, 30)$?

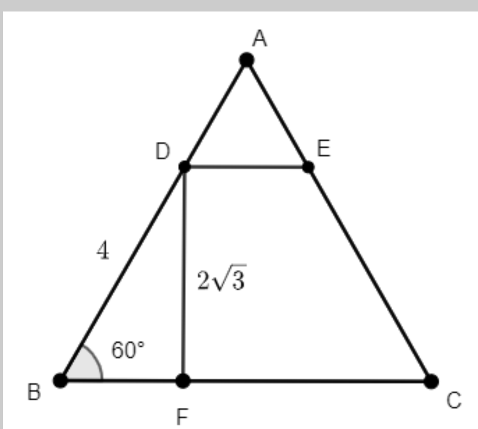
Solution: The slope of the segment is $\frac{30-3}{34-(-2)} = \frac{27}{36} = \frac{3}{4}$, meaning that every four units to the right and three units up there's another point on the segment. Since the start of the segment at $(-2, 3)$ is a lattice point, the next lattice point on the segment will be $(-2 + 4, 3 + 3) = (2, 6)$. Since you can repeat this process $(30 - 3) \div 3 = 9$ times from the first point, there will be a total of $\boxed{10}$ lattice points on the segment.

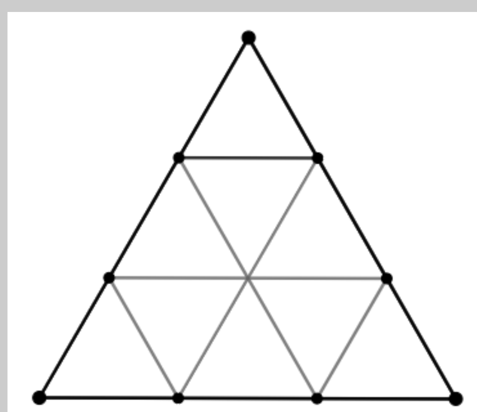
3. In equilateral triangle ABC with each side of length 6, a segment is drawn parallel to one side to form a trapezoid with legs of length 4. Find the area of the trapezoid.

Solution: By drawing segment $\overline{DE}$ and creating right triangle BDF, we know that $DE = 2$ because ΔADE is an equilateral triangle similar to ΔABC. Also, by creating a $30° - 60° - 90°$ triangle BDF, the height of the trapezoid is $2\sqrt{3}$. Using the trapezoid area formula,

$$A_{\text{trap}} = \frac{b_1 + b_2}{2}\cdot h = \frac{2+6}{2}2\sqrt{3} = \boxed{8\sqrt{3}}.$$

Alternatively, note that the triangle at the top is one-third the dimensions of the similar and original triangle so is therefore one-ninth of the area. Since the area of ΔABC is $\frac{6^2\sqrt{3}}{4} = 9\sqrt{3}$, the area of the trapezoid must be eight-ninths of that area or $8\sqrt{3}$.





4. Milo lists the first 100 positive integers, then crosses off every integer divisible by 4 or 5. What is the sum of the remaining integers?

Solution: Instead of determining the sum of the remaining integers, utilize the principle of inclusion-exclusion: find the sum of all, remove the sum of the ones you want to remove, then add back in anything you've doubly-removed. The sum of the first 100 positive integers is

$$\frac{100(1+100)}{2} = 50(101) = 5050.$$

The sum of the first 25 multiples of 4 is

$$\frac{25(4+100)}{2} = 25(52) = 100(13) = 1300.$$

The sum of the first 20 multiples of 5 is

$$\frac{20(5+100)}{2} = 10(105) = 1050.$$

We've doubly removed multiples of 20 (20, 40, 60, 80, 100) so we will add $20 + 40 + 60 + 80 + 100 = 300$ back in.

$$5050 - 1300 - 1050 + 300 = \boxed{3000}$$

5. Find xyz if

$$\begin{cases} 2x + 2y + z = 1 \\ x - y + 6z = 21 \\ 6x + 4y - 2z = -8 \end{cases}$$

Solution: There are many ways to accomplish this (substitution, elimination, matrices) but here we will representing the system as a 3 by 4 matrix and use row-reduction to gain the identity matrix on the left and the values of x, y, and z on the right. Here are some possible steps:

- $-(r_3 - 3r_1)$
- $(r_1 - r_3) \div 2$
- $r_2 - r_1$
- $(r_3 + 2r_2) \div 21$
- $(r_2 - 8r_3) \cdot -1$
- $r_1 + 2r_3$

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 1 & -1 & 6 & 21 \\ 6 & 4 & -2 & -8 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 1 & -1 & 6 & 21 \\ 0 & 2 & 5 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 1 & -1 & 6 & 21 \\ 0 & 2 & 5 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 2 & 5 & 11 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & -1 & 8 & 26 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We find $(x, y, z) = (1, -2, 3)$ and $xyz = \boxed{-6}$.

6. How many distinct three-letter words can be created from the letters in WOCOMAL?

Solution: First, given seven distinct letters, we can create $7 \cdot 6 \cdot 5 = 210$ distinct words from the letters in WOCOMAL. However, there are two Os. There are five groupings that can be made with two Os (one for each letter that isn't O) and $\frac{3!}{2!} = 3$ different ways to arrange those letters where there would normally be 6. We subtract $5 \cdot 3$ from 210 to arrive at $\boxed{195}$ unique words.

7. Evaluate the following expression if $x = 6$ and $y = 2$.

$$\frac{x^4 + x^3 - 6x^2}{xy^3 - 2y^3} \div \frac{x^2 + 8x + 15}{x^2 + 2x - 15} \cdot \frac{y^2}{x^3}$$

Solution: Instead of substituting first, factor and cancel and then evaluate:

$$\frac{x^4 + x^3 - 6x^2}{xy^3 - 2y^3} \div \frac{x^2 + 8x + 15}{x^2 + 2x - 15} \cdot \frac{y^2}{x^3} = \frac{x^2(x+3)(x-2)}{y^3(x-2)} \cdot \frac{(x+5)(x-3)}{(x+5)(x+3)} \cdot \frac{y^2}{x^3} = \frac{x-3}{xy}$$

Letting $x = 6$ and $y = 2$, the expression simplifies to $\frac{6-3}{2 \cdot 6} = \frac{3}{12} = \boxed{\frac{1}{4}}$

8. There are n concentric circles with radii $r_1, r_2, r_3, \ldots, r_n$ and areas $A, 2A, 3A, \ldots, nA$, respectively. Find the smallest value of n where r_n is less than 1% larger than r_{n-1}.

Solution: We see that since $A = \pi r_1^2$, $2A = \pi r_2^2$, $3A = \pi r_3^2$, that

$$r_1 = \sqrt{\frac{A}{\pi}} \qquad r_2 = \sqrt{\frac{2A}{\pi}} = \sqrt{2}\sqrt{\frac{A}{\pi}} \qquad r_3 = \sqrt{\frac{3A}{\pi}} = \sqrt{3}\sqrt{\frac{A}{\pi}}.$$

Since we want to find when

$$\frac{r_n}{r_{n-1}} = 1.01.$$

we replace the radii with

$$\frac{\sqrt{n}\sqrt{\frac{A}{\pi}}}{\sqrt{n-1}\sqrt{\frac{A}{\pi}}} = 1.01$$

$$\frac{\sqrt{n}}{\sqrt{n-1}} = 1.01$$

$$\frac{n}{n-1} = 1.0201$$

$$n = 1.0201n - 1.0201$$

$$1.0201 = .0201n$$

$$10201 = 201n$$

$$n = \frac{10201}{201}$$

Using long division, we find that 201 divides into 10201 just shy of 51 times (≈ 50.75 times), so n would need to equal $\boxed{51}$ for our initial goal to be realized.

9. Mr. Smithers comes home from the grocery store with three bags of groceries. Each bag has three distinct food items:

   - Bag 1: Vegetable, pasta, meat.
   - Bag 2: Vegetable, vegetable, meat.
   - Bag 3: Pasta, pasta, meat.

   He picks one item at random from each bag. What is the probability that he has picked one of each type of food (vegetable, pasta, meat)?

   Solution: For simplicity, let the bags contain the following items:

   - Bag 1: V, P, M
   - Bag 2: V_1, V_2, M
   - Bag 3: P_1, P_2, M

   There are $3 \cdot 3 \cdot 3 = 27$ different groups of three items that he can pick out. There are just eight possible groups that include three different items: $VMP_1, VMP_2, PV_1M, PV_2M, MV_1P_1, MV_2P_1, MV_1P_2, MV_2P_2$. This means the probability of Mr. Smithers cooking a balanced meal (vegetable, pasta, meat) is $\boxed{\frac{8}{27}}$.